

Statistical properties of a generalized threshold network model

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Abstract

The threshold network model is a type of finite random graphs. In this paper, we introduce a generalized threshold network model. A pair of vertices with random weights is connected by an edge when real-valued functions of the pair of weights belong to given Borel sets. We extend several known limit theorems for the number of prescribed subgraphs to show that the strong law of large numbers can be uniform convergence. We also prove two limit theorems for the local and global clustering coefficients.

1 Introduction

Complex networks have been an attractive research topic for a decade. Particularly, many real-world graphs are characterized by the small diameter, high clustering (abundance of connected triangles), and fat-tail degree distributions. Degree distributions often follow the truncated power law, which is called the scale-free property of networks [1, 4, 15]. Both growing and static network models are capable of generating scale-free networks.

Here we are concerned with asymptotic properties of a class of static network models called the threshold network model, which is generated on n vertices labeled $1, \dots, n$ with independent and identically distributed (i.i.d.) random weights X_1, \dots, X_n . We connect a pair of vertices i and j with $i \neq j$ by an edge when $X_i + X_j > \theta$ for a given threshold θ . The threshold network model is a subclass of so called hidden variable models and its mean behavior [5, 6, 8, 10, 11, 18, 19] and limit theorems [9, 14] have been analyzed.

To define a generalization of the threshold network model, let \mathbb{R}^d be the d -dimensional Euclidean space. We prepare an i.i.d. sequence of \mathbb{R}^d -valued random variables X_1, \dots, X_n with a common distribution function F . We associate the random variable X_i , which we call weight function, with vertex i . Now we introduce Borel measurable functions $f_c^{l'} : (\mathbb{R}^d)^2 \rightarrow \mathbb{R}$ with $f_c^{l'}(x, y) = f_c^{l'}(y, x)$ for all $l' \in \{1, \dots, l\}$. Let $\mathcal{B}(\mathbb{R})$ be the Borel σ -field of \mathbb{R} . For a given finite collection of Borel measurable sets $\mathcal{C} = \{B_1, \dots, B_l\}$ with $B_{l'} \in \mathcal{B}(\mathbb{R})$, we connect vertices i and j ($i \neq j$) if $f_c^{l'}(X_i, X_j) \in B_{l'}$ for all $l' \in \{1, \dots, l\}$. In other words, we form an edge $\langle i, j \rangle$ if $\prod_{l'=1}^l I_{B_{l'}}(f_c^{l'}(X_i, X_j)) = 1$ for $i \neq j$, where $I_A(x)$ denotes the indicator function, i.e., $I_A(x) = 1$ for $x \in A$ and $I_A(x) = 0$ otherwise. Thus we obtain a random graph $G_{\mathcal{C}}(X_1, \dots, X_n)$. If there exist two collections of Borel sets $\mathcal{C} = \{B_1, \dots, B_l\}$ and $\mathcal{C}' = \{B'_1, \dots, B'_l\}$ with $B_{l'} \subset B'_{l'}$ for all $l' \in \{1, \dots, l\}$, then $\mathbb{P}\{\langle i, j \rangle \in G_{\mathcal{C}}(X_1, \dots, X_n)\} \leq \mathbb{P}\{\langle i, j \rangle \in G_{\mathcal{C}'}(X_1, \dots, X_n)\}$ holds by simple coupling.

This random graph generalizes the threshold network model studied in [5, 6, 8–10, 12, 14]. By choosing $l = 1$, $B_1 = (\theta, \infty)$ for some $\theta \in \mathbb{R}$, $f_c^1(x, y) = x + y$, we reproduce the model in [5, 6, 8–10]. In the context of social networks, a model with $l = 2$, $B_1 = (\theta, \infty)$, $B_2 = (-\infty, c]$ ($\theta, c \in \mathbb{R}$), $f_c^1(x, y) = x + y$, and

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$f_c^2(x, y) = |x - y|$ (or $f_c^2(x, y) = |x - y|/(x + y)$) was proposed [12]. General limit theorems are shown in [14] when $l = 1$, $B_1 = (\theta, \infty)$, and $f_c^1(x, y) = |x - y|$.

In Sec. 2, we state several general limit theorems for the number of prescribed subgraphs. By using U -statistics, the strong law of large numbers, the central limit theorem, and the law of the iterated logarithm are stated for global properties of the model. We also state a limit distribution for a local property. These are generalizations of Theorems 1, 4, and 5 of [9] and Theorems 1, 2, and 3 of [14]. In Sec. 3, we show that the strong law of large numbers for the number of prescribed subgraphs is uniform convergence on so-called the VC class of Borel sets, generalizing Theorem 1a of [14]. In Sec. 4, we show limit theorems for the clustering coefficient, which quantifies the abundance of connected triangles in a graph in a specific ways. Particularly, we show the strong law of large numbers for the local clustering coefficient (Theorem 2) and the global clustering coefficient (Theorem 3). Theorems 2 and 3 are main results of this paper. In Sec. 5, we present several examples of limit degree distributions.

2 General Results

In this section, we show limit theorems for the number of prescribed subgraphs. Let us begin with notations [14]. For $m \in \{2, \dots, n\}$, we consider a graph $H = (V_H, E_H)$ on the ordered set of m vertices $V_H = (v_1, \dots, v_m)$ and the edge set E_H . For another graph $H' = (V_{H'}, E_{H'})$ on m vertices, we say $H' \sim H$ if $V_{H'} = V_H$ and $E_{H'} = E_H$ for some reordering of vertices. Thus $\mathcal{A}_H^\sim = \{H' : H' \sim H\}$ is the set of all graphs isomorphic to H . Let us define $\mathcal{A}_m = \bigcup_i \mathcal{A}_{H_i}^\sim$, where H_i is an arbitrarily chosen graph on m vertices. The collection of all triangles and graphs on three vertices that consist of two connected vertices and an isolated vertex is an example of \mathcal{A}_3 . The collection of cliques on m vertices and the graphs on m vertices with m isolated vertices is an example of \mathcal{A}_m . The definition of \mathcal{A}_m guarantees the symmetrical property of the kernel function $h_{\mathcal{A}_m} : (\mathbb{R}^d)^m \rightarrow \mathbb{R}$ given by

$$h_{\mathcal{A}_m}(x_1, \dots, x_m) = I_{\mathcal{A}_m}(G_{\mathcal{C}}(x_1, \dots, x_m)), \quad (1)$$

where $G_{\mathcal{C}}(x_1, \dots, x_m)$ denotes a realization of the random graph $G_{\mathcal{C}}(X_1, \dots, X_m)$. Then we define

$$\tilde{U}_n(\mathcal{C}, \mathcal{A}_m) = \sum_{1 \leq i_1 < \dots < i_m \leq n} h_{\mathcal{A}_m}(X_{i_1}, \dots, X_{i_m}),$$

i.e., the number of subgraphs belonging to the collection \mathcal{A}_m in the random graph $G_{\mathcal{C}}(X_1, \dots, X_n)$. We also define

$$U_n(\mathcal{C}, \mathcal{A}_m; i) = \sum_{\substack{1 \leq i_2 < \dots < i_m \leq n \\ i_2, \dots, i_m \neq i}} h_{\mathcal{A}_m}(X_i, X_{i_2}, \dots, X_{i_m}),$$

i.e., the number of subgraphs that include vertex i and belong to \mathcal{A}_m in the random graph $G_{\mathcal{C}}(X_1, \dots, X_n)$.

Note that $U_n(\mathcal{C}, \mathcal{A}_m; i)$, $1 \leq i \leq n$ are identical in distribution and the following relation holds:

$$\frac{\sum_{i=1}^n U_n(\mathcal{C}, \mathcal{A}_m; i) / \binom{n-1}{m-1}}{n} = \frac{m \tilde{U}_n(\mathcal{C}, \mathcal{A}_m)}{n \binom{n-1}{m-1}} = \frac{\tilde{U}_n(\mathcal{C}, \mathcal{A}_m)}{\binom{n}{m}}.$$

This implies that the global property $\tilde{U}_n(\mathcal{C}, \mathcal{A}_m) / \binom{n}{m}$ is the arithmetic mean of the local properties $U_n(\mathcal{C}, \mathcal{A}_m; 1) / \binom{n-1}{m-1}, \dots, U_n(\mathcal{C}, \mathcal{A}_m; n) / \binom{n-1}{m-1}$.

We define

$$F(\mathcal{C}, \mathcal{A}_m) = \mathbb{E}[h_{\mathcal{A}_m}(X_1, \dots, X_m)],$$

$$\zeta(\mathcal{C}, \mathcal{A}_m) = \text{Var}(\mathbb{E}[h_{\mathcal{A}_m}(X_1, \dots, X_m) | X_1]),$$

and assume $\zeta(\mathcal{C}, \mathcal{A}_m) > 0$. Since $\tilde{U}_n(\mathcal{C}, \mathcal{A}_m) / \binom{n}{m}$ is a U -statistic [17] obtained from the symmetric kernel $h_{\mathcal{A}_m}$, the strong law of large numbers (SLLN), the central limit theorem (CLT) and the law of the iterated logarithm (LIL) are derived from general results for the U -statistics, namely, Theorem A (SLLN) and Theorem B (LIL) in Section 5.4, and Theorem A (CLT) in Section 5.5 of [17]:

Fact 1 (SLLN for global property). As $n \rightarrow \infty$,

$$\frac{\tilde{U}_n(\mathcal{C}, \mathcal{A}_m)}{\binom{n}{m}} \rightarrow F(\mathcal{C}, \mathcal{A}_m), \quad \text{almost surely.}$$

Fact 2 (CLT for global property). As $n \rightarrow \infty$,

$$\sqrt{\frac{n}{m^2 \zeta(\mathcal{C}, \mathcal{A}_m)}} \left[\frac{\tilde{U}_n(\mathcal{C}, \mathcal{A}_m)}{\binom{n}{m}} - F(\mathcal{C}, \mathcal{A}_m) \right] \Rightarrow \mathcal{Z},$$

where \Rightarrow stands for convergence in distribution and \mathcal{Z} is a standard normal random variable.

Fact 3 (LIL for global property).

$$\limsup_{n \rightarrow \infty} \sqrt{\frac{n(\log \log n)^{-1}}{2m^2 \zeta(\mathcal{C}, \mathcal{A}_m)}} \left| \frac{\tilde{U}_n(\mathcal{C}, \mathcal{A}_m)}{\binom{n}{m}} - F(\mathcal{C}, \mathcal{A}_m) \right| = 1, \quad \text{almost surely.}$$

There are the direct generalization of Theorem 4 of [9] and Theorems 1, 2, and 3 of [14] to the present model.

Remark 1. Generally, when the $f_c^{l'}$ is asymmetric (e.g. directed graph), the number of graphs isomorphic to a graph H on m vertices is $\binom{n}{m} \cdot m!$. The limit theorems above are valid by replacing the normalizing factor $\binom{n}{m}$ with $\binom{n}{m} \cdot m!$.

By generalizing Theorems 1 and 5 of [9], we obtain the following asymptotic behavior:

Fact 4. As $n \rightarrow \infty$,

$$\frac{U_n(\mathcal{C}, \mathcal{A}_m; 1)}{\binom{n-1}{m-1}} \Rightarrow U(\mathcal{C}, \mathcal{A}_m),$$

where

$$U(\mathcal{C}, \mathcal{A}_m) = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} h_{\mathcal{A}_m}(X_1, x_2, \dots, x_m) F(dx_2) \cdots F(dx_m).$$

Remark 2. Almost sure convergence theorem for $U_n(\mathcal{C}, \mathcal{A}_m; 1)/\binom{n-1}{m-1}$ in the sense of Theorem 2 in Sec. 4.1 can be proved by a simple modification of the proof of Theorems 1 and 5 of [9].

3 Uniform Property

The Vapnik-Chervonenkis approach is well known in the context of the statistical learning theory. Particularly, it is useful in showing uniform convergence for limit theorems [7, 16]. In this section, we show that SLLN for global property (Fact 1) is uniform convergence on the VC class of the Borel sets, which extends the special case treated in [14].

Let M be a set and \mathcal{D} be a family of subsets of M . For $A \subset M$ let $\Delta^{\mathcal{D}}(A) = \sharp(A \cap \mathcal{D})$, where $\sharp(A \cap \mathcal{D})$ denotes the number of sets in $A \cap \mathcal{D} = \{A \cap D : D \in \mathcal{D}\}$. Let $m_{\mathcal{D}}(n) = \max_{A \subset M} \{\Delta^{\mathcal{D}}(A) : |A| = n\}$ for $n = 0, 1, 2, \dots$, where $|A|$ denotes the number of elements in A , or if $|M| < n$ let $m_{\mathcal{D}}(n) = m_{\mathcal{D}}(|M|)$. We define an indicator of the family \mathcal{D} :

$$S(\mathcal{D}) = \begin{cases} \sup \{n : m_{\mathcal{D}}(n) = 2^n\} & \text{if } \mathcal{D} \text{ is non-empty,} \\ -1 & \text{if } \mathcal{D} \text{ is empty.} \end{cases}$$

The family \mathcal{D} is called a Vapnik-Chervonenkis (VC) class of sets if $S(\mathcal{D}) < +\infty$. For example, the collection of half intervals $\mathcal{D} = \{(-\infty, x] : x \in \mathbb{R}\}$ is a VC class on \mathbb{R} with $S(\mathcal{D}) = 1$. Based on Chapter 4.5 of [7], we have

Corollary 1. For any $\mathcal{D} \subset 2^M$ and $\mathcal{D}' \subset 2^M$ (resp. 2^N), if $S(\mathcal{D}) < \infty$ and $S(\mathcal{D}') < \infty$ then $S(\mathcal{D} \cup \mathcal{D}') < \infty$ and $S(\mathcal{D} \cap \mathcal{D}') < \infty$ (resp. $S(\mathcal{D} \times \mathcal{D}') < \infty$) where $\mathcal{D} \cup \mathcal{D}' = \{D \cup D' : D \in \mathcal{D}, D' \in \mathcal{D}'\}$, $\mathcal{D} \cap \mathcal{D}' = \{D \cap D' : D \in \mathcal{D}, D' \in \mathcal{D}'\}$ and $\mathcal{D} \times \mathcal{D}' = \{D \times D' : D \in \mathcal{D}, D' \in \mathcal{D}'\}$.

For a given function $h : M \rightarrow \mathbb{R}$, the subgraph of h is the set

$$D_h = \{(x, t) \in M \times \mathbb{R} : 0 \leq t \leq h(x) \text{ or } h(x) \leq t \leq 0\}.$$

A class of functions \mathcal{H} is a VC-subgraph class if the collection $\mathcal{D}_{\mathcal{H}} = \{D_h : h \in \mathcal{H}\}$ is a VC class of sets.

For a class of real-valued measurable functions \mathcal{H} on M^m for a fixed integer m , Arcones and Giné [2] proved the following uniform SLLN for i.i.d. sequence $\{X_i\}_{i=1,2,\dots}$ on M :

Lemma 1 (Corollary 3.3 of [2]). *If \mathcal{H} is a measurable VC-subgraph class of functions with $\mathbb{E}[\sup_{h \in \mathcal{H}} |h(X_1, \dots, X_m)|] < +\infty$, then*

$$\sup_{h \in \mathcal{H}} \left[\frac{1}{\binom{n}{m}} \sum_{1 \leq i_1 < \dots < i_m \leq n} h(X_{i_1}, \dots, X_{i_m}) - \mathbb{E}[h(X_1, \dots, X_m)] \right] \rightarrow 0, \text{ almost surely,}$$

as $n \rightarrow \infty$.

In order to use Lemma 1, we rewrite the kernel function (1) and show that the family of the kernel functions indexed by the VC class of Borel sets is a VC-subgraph class. We assume $d = l = 1$. For a fixed integer $m \geq 2$ and $f_c^1 : \mathbb{R}^2 \rightarrow \mathbb{R}$, we define a function $G : \mathbb{R}^m \rightarrow \mathbb{R}^{\binom{m}{2}}$ by

$$G(x_1, \dots, x_m) = (f_c^1(x_1, x_2), f_c^1(x_1, x_3), \dots, f_c^1(x_1, x_m), f_c^1(x_2, x_3), \dots, f_c^1(x_{m-1}, x_m)).$$

Each coordinate corresponds to a potential edge of the graph with a lexicographic order. For example, if $m = 4$, the first coordinate corresponds to the $\langle 1, 2 \rangle$, the second coordinate to $\langle 1, 3 \rangle$, and the sixth coordinate to $\langle 3, 4 \rangle$. Note that edge $\langle i, j \rangle$ exists if and only if $f_c^1(x_i, x_j) \in B$ for a given Borel set B .

For a given collection \mathcal{A}_m of graphs on m vertices, we define a set $\tilde{\mathcal{A}}_m$ on $\mathbb{R}^{\binom{m}{2}}$ as follows. For each graph $H \in \mathcal{A}_m$, we associate a set \tilde{H} on $\mathbb{R}^{\binom{m}{2}}$. When a pair of vertices in H has an edge, the corresponding coordinate of \tilde{H} is occupied by B . Otherwise, it is occupied by B^c , where B^c denotes the complement of B . For example, if $m = 4$ and H has edge set $\{\langle 1, 2 \rangle, \langle 2, 3 \rangle\}$, then $\tilde{H} = B \times B^c \times B^c \times B \times B^c \times B^c$. Then we define the set $\tilde{\mathcal{A}}_m = \bigcup_{H \in \mathcal{A}_m} \tilde{H}$. Note that $G(x_1, \dots, x_m) \in \tilde{\mathcal{A}}_m$ is equivalent to the event that the realization of the random graph with weights x_1, \dots, x_m is in \mathcal{A}_m . Finally, we obtain the rewritten form of the kernel function:

$$h_{\mathcal{A}_m}^B(x_1, \dots, x_m) = I_{\tilde{\mathcal{A}}_m}(G(x_1, \dots, x_m)) = I_{G^{-1}(\tilde{\mathcal{A}}_m)}(x_1, \dots, x_m), \quad (2)$$

where $G^{-1}(\tilde{\mathcal{A}}_m)$ denotes the inverse image of $\tilde{\mathcal{A}}_m$.

Let \mathcal{D} be a VC class of Borel sets on \mathbb{R} . For a fixed collection \mathcal{A}_m , we consider the class of kernel functions $\mathcal{H}_{\mathcal{D}} = \{h_{\mathcal{A}_m}^B(x_1, \dots, x_m) : B \in \mathcal{D}\}$. By Corollary 1, if \mathcal{D} is a VC class of sets in general, then the class of indicators $\{I_D : D \in \mathcal{D}\}$ is a VC-subgraph class. From Eq. (2), if the collection $G_{\mathcal{D}}^{-1}(\tilde{\mathcal{A}}_m) = \{G^{-1}(\tilde{\mathcal{A}}_m) : B \in \mathcal{D}\}$ is a VC class then $\mathcal{H}_{\mathcal{D}}$, which is a collection of indicator functions, is a VC-subgraph class on \mathbb{R}^m . Indeed, $G_{\mathcal{D}}^{-1}(\tilde{\mathcal{A}}_m)$ is a VC class based on Theorem 4.2.3 of [7] and Corollary 1 above, and $\mathcal{H}_{\mathcal{D}}$ is a VC-subgraph class. Finally, we find the uniform version of Fact 1 by Lemma 1:

Theorem 1. *If \mathcal{D} be a VC class of Borel sets, then for a fixed f_c and \mathcal{A}_m ,*

$$\sup_{B \in \mathcal{D}} \left[\frac{1}{\binom{n}{m}} \sum_{1 \leq i_1 < \dots < i_m \leq n} h_{\mathcal{A}_m}^B(X_{i_1}, \dots, X_{i_m}) - \mathbb{E}[h_{\mathcal{A}_m}^B(X_1, \dots, X_m)] \right] \rightarrow 0,$$

almost surely, as $n \rightarrow \infty$.

Theorem 1 can be extended to general d and l . It generalizes the uniform SLLN in Theorem 1a of [14], which deals with the collection of half intervals as a VC class of sets.

4 Clustering Coefficient

Real-world networks are often equipped with high clustering, that is, a large number of connected triangles. The clustering coefficient quantifies the density of triangles in a graph (see [1, 15] for review). In this section, we study the limit theorems for the clustering coefficient.

4.1 Local Clustering Coefficient

We assume $d = l = 1$; extensions of the following results to general d and l is straightforward. We consider a random graph $G_B(X_1, \dots, X_n)$ for a given Borel set B and $f_c \equiv f_c^1$. Then we define

$$D_n(i) = \sum_{\substack{1 \leq j \leq n \\ j \neq i}} h_D(X_i, X_j),$$

$$T_n(i) = \sum_{\substack{1 \leq j < k \leq n, \\ j, k \neq i}} h_T(X_i, X_j, X_k),$$

where $h_D(x, y) = I_B(f_c(x, y))$ and $h_T(x, y, z) = I_B(f_c(x, y)) \cdot I_B(f_c(y, z)) \cdot I_B(f_c(z, x))$, i.e., $D_n(i)$ is the degree of vertex i and $T_n(i)$ is the number of triangles including vertex i . The local clustering coefficient $C_n(i)$ of vertex i is given by

$$C_n(i) = \frac{T_n(i)}{\binom{D_n(i)}{2}} \cdot I_{\{D_n(i) \geq 2\}} + w \cdot I_{\{D_n(i) = 0, 1\}}$$

for an indeterminate w . The second term represents the singular part for which the local clustering coefficient is not defined in physics literature and applications. Here we retain this term to assess the contribution of vertices with degree 0 or 1. If it is necessary to restrict $C_n(i) \in [0, 1]$, we must substitute a real value on $[0, 1]$ into w . If we substitute 0 into w , the contribution of these vertices to $C_n(i)$ is ignored. If we substitute 1, this contribution is implied to be the maximum because vertices with degree more than one satisfies $C_n(i) \leq 1$. Now we define

$$V_n(i) = \sum_{\substack{1 \leq j < k \leq n, \\ j, k \neq i}} h_V(X_i, X_j, X_k),$$

where $h_V(x, y, z) = I_B(f_c(x, y)) \cdot I_B(f_c(x, z))$, which represents the number of vertex pairs (j, k) such that both vertex j and vertex k are connected to vertex i . We note the relation: On $\{D_n(i) \geq 2\}$ or equivalently $\{V_n(i) \geq 1\}$,

$$\binom{D_n(i)}{2} = V_n(i),$$

which leads to

$$C_n(i) = \frac{T_n(i)}{V_n(i)} \cdot I_{\{V_n(i) \geq 1\}} + w \cdot I_{\{V_n(i) = 0\}}.$$

We also define

$$C(i) = \frac{E_T(X_i)}{E_D(X_i)^2} \cdot I_{\{E_D(X_i) > 0\}} + w \cdot I_{\{E_D(X_i) = 0\}}, \quad (3)$$

where

$$E_D(X_i) = \int_{\mathbb{R}} h_D(X_i, y) F(dy),$$

$$E_T(X_i) = \int_{\mathbb{R}} \int_{\mathbb{R}} h_T(X_i, y, z) F(dy) F(dz).$$

We consider $C_n(i; x)$, $C(i; x)$, $D_n(i; x)$, $T_n(i; x)$, and $V_n(i; x)$, which are random variables $C_n(i)$, $C(i)$, $D_n(i)$, $T_n(i)$ and $V_n(i)$ restricted to the subspace such that $\{X_i = x\}$. For example, $T_n(1; x) = \sum_{2 \leq j < k \leq n} h_T(x, X_j, X_k)$. We obtain the following asymptotic results for $C_n(i)$:

Theorem 2. As $n \rightarrow \infty$,

(i) For any $x \in \mathbb{R}$, $C_n(1; x) \rightarrow C(1; x)$, almost surely.

In particular,

(ii) $C_n(1) \rightarrow C(1)$, almost surely.

Proof. For an arbitrary fixed $x \in \mathbb{R}$, we first prove

$$\mathbb{E}[h_V(x, X_2, X_3)] = 0 \iff \mathbb{P}(V_n(1; x) = 0 \text{ for all } n \geq 1) = 1. \quad (4)$$

Indeed, if $\mathbb{E}[h_V(x, X_2, X_3)] = 0$ then

$$\mathbb{E}[V_n(1; x)] = \mathbb{E}\left[\sum_{2 \leq j < k \leq n} h_V(x, X_j, X_k)\right] = \sum_{2 \leq j < k \leq n} \mathbb{E}[h_V(x, X_j, X_k)] = 0$$

for all $n \geq 1$. Conversely, if $\mathbb{E}[V_n(1; x)] = 0$ for all $n \geq 1$ then

$$\mathbb{E}[h_V(x, X_2, X_3)] = \mathbb{E}[V_3(1; x)] = 0.$$

Therefore, we obtain

$$\mathbb{E}[h_V(x, X_2, X_3)] = 0 \iff \mathbb{E}[V_n(1; x)] = 0 \text{ for all } n \geq 1.$$

Since $V_n(1; x)$ is nonnegative,

$$\mathbb{E}[V_n(1; x)] = 0 \iff \mathbb{P}(V_n(1; x) = 0) = 1 \text{ for all } n \geq 1.$$

Moreover, $\{V_n(1; x) = 0\}$ is nonincreasing with n , which implies

$$\mathbb{P}(V_n(1; x) = 0) = 1 \text{ for all } n \geq 1 \iff \mathbb{P}(V_n(1; x) = 0 \text{ for all } n \geq 1) = 1.$$

Thus we have Eq. (4). By definition, $V_n(1; x)$ is invariant under any permutation on $\{x_2, x_3, \dots, x_n\}$, and $V_n(1; x)$ is nondecreasing, i.e.,

$$V_n(1; x)(x_2, x_3, \dots, x_n) \leq V_{n+1}(1; x)(x_2, x_3, \dots, x_n, x_{n+1}) \quad (5)$$

for all $n \geq 1$. Therefore $\mathbb{P}(V_n(1; x) = 0 \text{ for all } n \geq 1)$ equals to zero or one by the Hewitt-Savage zero-one law (see Theorem 36.5 of [3]). So we have

$$\mathbb{E}[h_V(x, X_2, X_3)] > 0 \iff \mathbb{P}(V_n(1; x) = 0 \text{ for all } n \geq 1) = 0 \iff \mathbb{P}(V_n(1; x) \geq 1 \text{ for some } n \geq 1) = 1.$$

Using Eq. (5), $\{V_n(1; x) \geq 1 \text{ for some } n \geq 1\}$ is equivalent to the event $\{\exists N \geq 1 \text{ s.t. } V_n(1; x) \geq 1 \text{ for all } n \geq N\}$. Hence we obtain

$$\begin{aligned} \mathbb{E}[h_V(x, X_2, X_3)] > 0 &\iff \mathbb{P}(\exists N \geq 1 \text{ s.t. } V_n(1; x) \geq 1 \text{ for all } n \geq N) = 1 \\ &\iff \mathbb{P}(\exists N \geq 1 \text{ s.t. } C_n(1; x) = T_n(1; x)/V_n(1; x) \text{ for all } n \geq N) = 1. \end{aligned} \quad (6)$$

Since $h_T(x, x_2, x_3)$ and $h_V(x, x_2, x_3)$ are symmetric functions of x_2 and x_3 , we define U -statistics

$$\begin{aligned} \frac{T_n(1; x)}{\binom{n-1}{2}} &= \frac{1}{\binom{n-1}{2}} \sum_{2 \leq j < k \leq n} h_T(x, X_j, X_k), \\ \frac{V_n(1; x)}{\binom{n-1}{2}} &= \frac{1}{\binom{n-1}{2}} \sum_{2 \leq j < k \leq n} h_V(x, X_j, X_k). \end{aligned}$$

We have the following SLLN by Theorem A in Section 5.4 of [17]: As $n \rightarrow \infty$,

$$\frac{T_n(1; x)}{\binom{n-1}{2}} \rightarrow \mathbb{E}[h_T(x, X_2, X_3)], \quad \text{almost surely,} \quad (7)$$

$$\frac{V_n(1; x)}{\binom{n-1}{2}} \rightarrow \mathbb{E}[h_V(x, X_2, X_3)], \quad \text{almost surely.} \quad (8)$$

Based on Eqs. (7) and (8), the corresponding clustering coefficient

$$C_n(1; x) = \frac{T_n(1; x)}{V_n(1; x)} = \frac{T_n(1; x) / \binom{n-1}{2}}{V_n(1; x) / \binom{n-1}{2}}$$

converges to $\mathbb{E}[h_T(x, X_2, X_3)] / \mathbb{E}[h_V(x, X_2, X_3)]$, almost surely as $n \rightarrow \infty$. By Eq. (6), we have

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} C_n(1; x) = \mathbb{E}[h_T(x, X_2, X_3)] / \mathbb{E}[h_V(x, X_2, X_3)]\right) = 1. \quad (9)$$

On the other hand, Eqs. (4) and (9) imply that $\mathbb{E}[h_V(x, X_2, X_3)] = 0$ is equivalent to $\mathbb{P}(\lim_{n \rightarrow \infty} C_n(1; x) = w) = 1$. With the relation $\mathbb{E}[h_V(x, X_2, X_3)] = \mathbb{E}[h_D(x, X_2)]^2$, we obtain

$$C_n(1; x) \rightarrow C(1; x) = \frac{\mathbb{E}[h_T(x, X_2, X_3)]}{\mathbb{E}[h_D(x, X_2)]^2} \cdot I_{\{\mathbb{E}[h_D(x, X_2)] > 0\}} + w \cdot I_{\{\mathbb{E}[h_D(x, X_2)] = 0\}},$$

almost surely as $n \rightarrow \infty$. Particularly, we have by using Fubini's theorem,

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} C_n(1) = C(1)\right) = \int_{\mathbb{R}} \mathbb{P}\left(\lim_{n \rightarrow \infty} C_n(1; x) = C(1; x)\right) F(dx) = \int_{\mathbb{R}} 1 \cdot F(dx) = 1.$$

This completes the proof. \square

4.2 Global Clustering Coefficient

The global clustering coefficient is defined by

$$C_n = \frac{1}{n} \sum_{i=1}^n C_n(i).$$

Since it is a symmetric function of (x_1, \dots, x_n) , we can prove SLLN for C_n by using the ergodic theory.

Theorem 3. *As $n \rightarrow \infty$,*

$$C_n \rightarrow \mathbb{E}[C(1)], \quad \text{almost surely.}$$

Proof. For simplicity, we only deal with the case $\mathbb{E}[C(1)] = 0$. For general cases, we can prove the theorem by replacing $C(1)$ by $C(1) - \mathbb{E}[C(1)]$. Let $\mathbf{x} = (x_1, x_2, \dots)$ be an infinite vector and $\mathbf{x}_k = x_k$. We define measure-preserving transformation T_n for the product measure \mathbb{P} by

$$(T_n \mathbf{x})_k = \begin{cases} x_{k+1} & \text{if } 1 \leq k \leq n-1, \\ x_1 & \text{if } k = n, \\ x_k & \text{otherwise,} \end{cases}$$

for each $n \geq 1$. By denoting $C_n(i; \mathbf{x}) = C_n(i; x_i)$, a realization of C_n is represented by

$$C_n(\mathbf{x}) = \frac{1}{n} \sum_{i=0}^{n-1} C_n(1; T_n^i \mathbf{x}).$$

For arbitrary fixed $\varepsilon > 0$, we define

$$\begin{aligned} C_n^\varepsilon(1; \mathbf{x}) &= (C_n(1; \mathbf{x}) - \varepsilon) \cdot I_{A_\varepsilon}, \\ S_n^\varepsilon(\mathbf{x}) &= \sum_{i=0}^{n-1} C_n^\varepsilon(1; T_n^i \mathbf{x}), \end{aligned} \quad (10)$$

where $A_\varepsilon = \{\mathbf{x} : \limsup_{n \rightarrow \infty} C_n(\mathbf{x}) > \varepsilon\}$. Using the maximal ergodic theorem (see Theorem 24.2 of [3]),

$$\int_{M_n^\varepsilon} C_n^\varepsilon(1; \mathbf{x}) d\mathbb{P} \geq 0$$

for every $n \geq 1$, where $M_n^\varepsilon = \{\mathbf{x} : \sup_{1 \leq j \leq n} S_j^\varepsilon(\mathbf{x}) > 0\}$. On the other hand, we have

$$M_n^\varepsilon \uparrow \left\{ \mathbf{x} : \sup_{k \geq 1} S_k^\varepsilon(\mathbf{x}) > 0 \right\} = \left\{ \mathbf{x} : \sup_{k \geq 1} \frac{S_k^\varepsilon(\mathbf{x})}{k} > 0 \right\} = \left\{ \mathbf{x} : \sup_{k \geq 1} C_k(\mathbf{x}) > \varepsilon \right\} \cap A_\varepsilon = A_\varepsilon,$$

as $n \rightarrow \infty$ by Eq. (10). From the dominated convergence theorem and Theorem 2, we derive

$$0 \leq \int_{M_n^\varepsilon} C_n^\varepsilon(1; \mathbf{x}) d\mathbb{P} \rightarrow \int_{A_\varepsilon} [C(1; \mathbf{x}) - \varepsilon] d\mathbb{P}, \quad (11)$$

as $n \rightarrow \infty$.

Let \mathcal{I}_n be the class of sets that are invariant under all permutations of the first n coordinates and $\mathcal{I} = \bigcap_{n=1}^\infty \mathcal{I}_n$. It is easy to check that $A_\varepsilon \in \mathcal{I}$. Since $\mathbb{P}(A)$ equals to zero or one for any $A \in \mathcal{I}$ by the Hewitt-Savage zero-one law (see Theorem 36.5 of [3]), the conditional expectation $\mathbb{E}[C(1)|\mathcal{I}]$ equals to $\mathbb{E}[C(1)]$, almost surely. This leads to

$$\begin{aligned} 0 &\leq \int_{A_\varepsilon} [C(1; \mathbf{x}) - \varepsilon] d\mathbb{P} = \int_{A_\varepsilon} C(1; \mathbf{x}) d\mathbb{P} - \varepsilon \mathbb{P}(A_\varepsilon) = \int_{A_\varepsilon} \mathbb{E}[C(1; \mathbf{x})|\mathcal{I}] d\mathbb{P} - \varepsilon \mathbb{P}(A_\varepsilon) \\ &= \int_{A_\varepsilon} \mathbb{E}[C(1)] d\mathbb{P} - \varepsilon \mathbb{P}(A_\varepsilon) = -\varepsilon \mathbb{P}(A_\varepsilon), \end{aligned}$$

by Eq. (11) and $\mathbb{E}[C(1)] = 0$. Then we have $\mathbb{P}(A_\varepsilon) = 0$ for any $\varepsilon > 0$ and therefore $\limsup_{n \rightarrow \infty} C_n \leq 0$, almost surely. Repeating the same argument for $-C_n$, we have $\liminf_{n \rightarrow \infty} C_n \geq 0$, almost surely. This completes the proof. \square

Here we show a simple example for Theorem 3. Consider an i.i.d. sequence X_1, \dots, X_n such that $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = 0) = 1 - p$ for all $i = 1, \dots, n$. Let $B_1 = (\theta, \infty)$ and $f_c(x, y) = x + y$. We set a threshold θ such that $0 < \theta < 1$. In this case, a pair of vertices i and j with $i \neq j$ is disconnected if and only if $X_i = X_j = 0$. By direct computation, we have

$$\mathbb{E}[C(1)] = p \cdot C(1; 1) + (1 - p) \cdot C(1; 0) = p \cdot \frac{p^2 + 2p(1 - p)}{1} + (1 - p) \cdot \frac{p^2}{p^2} = 1 - p(1 - p)^2.$$

In order to calculate C_n , let $S_n = \sum_{i=1}^n X_i$, that is, the number of vertices with $X_i = 1$. We use the symbols x_i , s_n and c_n as realization of random variables X_i , S_n and C_n respectively. If $s_n = 0$, the graph consists of n isolated vertices. In this case $c_n = w$. If $s_n = 1$, the graph is the star in which only one central vertex has $n - 1$ edges and other $n - 1$ vertices are connected only to the center. So we obtain

$$c_n = \frac{1}{n} \{0 \cdot 1 + w \cdot (n - 1)\} = \left(1 - \frac{1}{n}\right) \cdot w.$$

If $2 \leq s_n \leq n - 2$, s_n vertices with $x_i = 1$ have $n - 1$ edges, and the other $n - s_n$ vertices are connected only to the vertices with $x_i = 1$. So we have

$$c_n = \frac{1}{n} \left\{ \frac{\binom{n-1}{2} - \binom{n-s_n}{2}}{\binom{n-1}{2}} \cdot s_n + \frac{\binom{s_n}{2}}{\binom{s_n}{2}} \cdot (n - s_n) \right\} = 1 - \frac{(n - s_n)(n - 1 - s_n)s_n}{n(n - 1)(n - 2)}.$$

If $s_n = n - 1$ or n , we obtain the complete graph, and $c_n = 1$. Noting

$$1 - \frac{(n - S_n)(n - 1 - S_n)S_n}{n(n - 1)(n - 2)} = \begin{cases} 1 & \text{if } s_n = 0, n - 1, n, \\ 1 - \frac{1}{n} & \text{if } s_n = 1, \\ 1 - \frac{(n - s_n)(n - 1 - s_n)s_n}{n(n - 1)(n - 2)} & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} C_n &= \left[1 - \frac{(n - S_n)(n - 1 - S_n)S_n}{n(n - 1)(n - 2)} \right] \cdot \{I_{\{2, \dots, n\}}(S_n) + r \cdot I_{\{0, 1\}}(S_n)\} \\ &= \left[1 - \left(1 - \frac{S_n}{n}\right) \left(1 - \frac{S_n}{n - 1}\right) \left(\frac{S_n}{n - 2}\right) \right] \cdot \{1 + (r - 1) \cdot I_{\{0, 1\}}(S_n)\} \\ &\rightarrow 1 - (1 - p)^2 p = \mathbb{E}[C(1)], \quad \text{almost surely } (n \rightarrow \infty). \end{aligned}$$

The last convergence comes from SLLN for the i.i.d. sequence.

One of our motivations to study limit theorems for the clustering coefficients is to make a clear distinction between the proportion of triangles in an entire graph and the clustering coefficients. By Eq. (66) of [9], the normalized number of triangles including vertex 1 converges to $E_T(x)$, almost surely for each realization x of X_1 , where the normalization constant is equal to $\binom{n-1}{2}$. In the same way, the degree of vertex 1 normalized by $n-1$ converges to $E_D(x)$, almost surely. Thus, when $E_D(x) > 0$, the local clustering coefficient converges almost surely to $E_T(x)/E_D(x)^2$, that is, the limit of the normalized number of triangles divided by the square of the limit of the normalized degree. The mean field result corresponding to Theorem 2 is found in Eq. (3) of [18]. The denominator equals to $E_T(x)$ and the numerator $[k(x)/N]^2$ converges to $E_D(x)^2$, almost surely as $N \rightarrow \infty$, where $k(x)$ is the degree of the vertex 1 and N denotes the number of vertices. Equation (30) of [5] corresponds to the normalized number of triangles. These heuristic results are consistent with our rigorous result. Several examples for the global clustering coefficient are calculated in [10].

In practice, we may substitute 0 into w and consider

$$\begin{aligned}\tilde{C}_n &= \frac{1}{n - \text{number of vertices with degree 0 or 1}} \sum_{i=1}^n C_n(i) \\ &= \frac{1}{n - \sum_{i=1}^n I_{\{0\}}(V_n(i))} \sum_{i=1}^n C_n(i),\end{aligned}$$

instead of C_n . Using the same arguments of Theorems 2 and 3, it is easy to prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n I_{\{0\}}(V_n(i)) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n I_{\{w\}}(C_n(i)) = \mathbb{P}(C(1) = w) = \mathbb{P}(E_D(X_1) = 0), \quad \text{almost surely.}$$

The last equality follows from the definition of $C(1)$, i.e., Eq. (3). Noting that

$$\tilde{C}_n = \frac{1}{1 - (1/n) \sum_{i=1}^n I_{\{0\}}(V_n(i))} \cdot C_n,$$

we have the following:

Corollary 2. *As $n \rightarrow \infty$,*

$$\tilde{C}_n \rightarrow \frac{1}{1 - \mathbb{P}(E_D(X_1) = 0)} \cdot \mathbb{E}[C(1)], \quad \text{almost surely.}$$

5 Examples

In this section, we show examples of the limit degree distribution, i.e., $m = 2$ and \mathcal{A}_2 is chosen as the collection of all possible edges in the limit theorem (Fact 4). We consider the case $l = 1$ with $f_c \equiv f_c^1$. We assume that the random variable X_1 is absolutely continuous so that it has a probability density function f . Let $\text{supp } f = \{x \in \mathbb{R} : f(x) \neq 0\}$ be the support of f .

We first set $B_1 = (\theta, \infty)$ and $f_c(x, y) = x + y$, i.e., the threshold network model in which an edge $\langle i, j \rangle$ forms if $\theta < X_i + X_j$ for a given threshold $\theta \in \mathbb{R}$ [5, 6, 9, 10]. By calculating the characteristic function of $D = U(B_1, \mathcal{A}_2)$, namely, the density of edges connected to vertex 1, we obtain the following results:

Lemma 2. *1. If there exists $a \in \mathbb{R}$ such that $\text{supp } f = [a, \infty)$, then*

$$D \sim \begin{cases} \delta_1(dk) & \text{if } \theta \leq 2a, \\ I_{(1-F(\theta-a), 1)}(k) \cdot \frac{f(\theta - F^{-1}(1-k))}{f(F^{-1}(1-k))} dx & \\ + (1 - F(\theta - a)) \cdot \delta_1(dk) & \text{if } \theta > 2a. \end{cases}$$

2. If there exists $b \in \mathbb{R}$ such that $\text{supp } f = (-\infty, b]$, then

$$D \sim \begin{cases} (1 - F(\theta - b)) \cdot \delta_0(dk) \\ + I_{(0, 1-F(\theta-b))}(k) \cdot \frac{f(\theta - F^{-1}(1-k))}{f(F^{-1}(1-k))} dk & \text{if } \theta < 2b, \\ \delta_0(dk) & \text{if } \theta \geq 2b. \end{cases}$$

3. If there exist $a, b \in \mathbb{R}$ such that $\text{supp } f = [a, b]$, then

$$D \sim \begin{cases} \delta_1(dk) & \text{if } \theta \leq 2a, \\ I_{(1-F(\theta-a), 1)}(k) \cdot \frac{f(\theta - F^{-1}(1-k))}{f(F^{-1}(1-k))} dk \\ + (1 - F(\theta - a)) \cdot \delta_1(dk) & \text{if } 2a < \theta < a + b, \\ I_{(0, 1)}(k) \cdot \frac{f(a+b - F^{-1}(1-k))}{f(F^{-1}(1-k))} dk & \text{if } \theta = a + b, \\ (1 - F(\theta - b)) \cdot \delta_0(dk) \\ + I_{(0, 1-F(\theta-b))}(k) \cdot \frac{f(\theta - F^{-1}(1-k))}{f(F^{-1}(1-k))} dk & \text{if } a + b < \theta < 2b, \\ \delta_0(dk) & \text{if } \theta \geq 2b. \end{cases}$$

Furthermore, if f is symmetric on $\text{supp } f$, then

$$D \sim I_{(0, 1)}(k) dk \quad \text{if } \theta = a + b.$$

4. If $\text{supp } f = (-\infty, \infty)$, then

$$D \sim I_{(0, 1)}(k) \cdot \frac{f(\theta - F^{-1}(1-k))}{f(F^{-1}(1-k))} dk$$

for any $\theta \in \mathbb{R}$.

Example 1. (Exponential distribution) If the random variable X_1 has the probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{otherwise,} \end{cases} \quad (12)$$

for a given $\lambda > 0$, then

$$D \sim \begin{cases} \delta_1(dk) & \text{if } \theta \leq 0, \\ I_{(e^{-\lambda\theta}, 1)}(k) \cdot \frac{e^{-\lambda\theta}}{k^2} dk + e^{-\lambda\theta} \cdot \delta_1(dk) & \text{if } \theta > 0. \end{cases}$$

Example 2. (Pareto distribution) If

$$f(x) = \begin{cases} \frac{c}{a} \cdot \left(\frac{a}{x}\right)^{c+1} & \text{if } x \geq a, \\ 0 & \text{otherwise,} \end{cases}$$

for given $a, c > 0$, then

$$D \sim \begin{cases} \delta_1(dk) & \text{if } \theta \leq 2a, \\ I_{((\frac{a}{\theta-a})^c, 1)}(k) \cdot \left(\frac{a}{\theta \cdot k^{1/c-a}}\right)^{c+1} dk + \left(\frac{a}{\theta-a}\right)^c \cdot \delta_1(dk) & \text{if } \theta > 2a. \end{cases}$$

The distribution of D of these two examples is proportional to $k^{-\alpha}$. The exponent α equals 2 in Example 1 and $1 + 1/c$ in Example 2. Because of a lower cutoff of f in both examples, the limit distributions have weights on δ_1 .

Example 3. (*Uniform distribution*) If

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$D \sim \begin{cases} \delta_1(dk) & \text{if } \theta \leq 0, \\ I_{(1-\theta,1)}(k)dk + (1-\theta) \cdot \delta_1(dk) & \text{if } 0 < \theta < 1, \\ I_{(0,1)}(k)dk & \text{if } \theta = 1, \\ (\theta-1) \cdot \delta_0(dk) + I_{(0,2-\theta)}(k)dk & \text{if } 1 < \theta < 2, \\ \delta_0(dk) & \text{if } \theta \geq 2. \end{cases}$$

In this case, the limit distribution is mixture of the uniform distribution and the delta measure.

By choosing $B_1 = (\theta_1, \theta_2]$ and $f_c(x, y) = x + y$, we obtain a generalization of the model investigated in [5, 6, 9, 10]. More precisely, an edge $\langle i, j \rangle$ forms if $\theta_1 < X_i + X_j \leq \theta_2$ for given thresholds $\theta_1, \theta_2 \in \mathbb{R}$ such that $\theta_1 < \theta_2$. To calculate the characteristic function of $D = U(B_1, \mathcal{A}_2)$, we consider the case in which a random variable X_1 has the probability density function (12), i.e., the exponential distribution, for which the limit distribution is represented by:

$$D \sim \begin{cases} \delta_0(dk) & \text{if } \theta_1 < \theta_2 \leq 0, \\ e^{-\lambda\theta_2} \cdot \delta_0(dk) \\ + I_{(0, 1-e^{-\lambda\theta_2})}(k) \cdot \frac{e^{-\lambda\theta_2}}{(1-k)^2} dk & \text{if } \theta_1 \leq 0 < \theta_2, \\ e^{-\lambda\theta_2} \cdot \delta_0(dk) \\ + I_{(0, 1-e^{-\lambda(\theta_2-\theta_1)})}(k) \cdot \frac{e^{-\lambda\theta_2}}{(1-k)^2} dk \\ + I_{(e^{-\lambda\theta_1}-e^{-\lambda\theta_2}, 1-e^{-\lambda(\theta_2-\theta_1)})}(k) \cdot \frac{e^{-\lambda\theta_1}-e^{-\lambda\theta_2}}{k^2} dk & \text{if } 0 < \theta_1 < \theta_2. \end{cases}$$

Finally, we deal with an example with $l = 2$. For fixed $\theta \in \mathbb{R}$ and $c \in [0, \infty)$, we choose $B_1 = (\theta, \infty]$, $B_2 = (0, c]$, $f_c^1(x, y) = x + y$, and $f_c^2(x, y) = |x - y|$. We consider the case in which X_1 is distributed according to the exponential distribution (Eq. (12)). This is the model proposed in [12]. Because the kernel function of this model is

$$h(x, x_2) = \begin{cases} I_{[-c+x, c+x]}(x_2) & \text{if } \frac{\theta+c}{2} \leq x, \\ I_{(\theta-x, c+x]}(x_2) & \text{if } \frac{\theta-c}{2} \leq x \leq \frac{\theta+c}{2}, \\ 0 & \text{if } x \leq \frac{\theta-c}{2}, \end{cases}$$

the limit distribution $D = U(\mathcal{C}_{\theta, c}, \mathcal{A}_2)$ is the following:

$$D \sim \begin{cases} (1 - e^{-\lambda(\theta-c)/2}) \delta_0(dk) \\ + I_{(0, 2e^{-\lambda(\theta+c)/2} \sinh(\lambda c)]}(k) \cdot g(k)dk & \text{if } c \leq \theta, \\ I_{(0, e^{-\lambda\theta}-e^{-\lambda c}]}(k) \cdot \frac{1}{2 \sinh(\lambda c)} dk \\ + I_{(e^{-\lambda\theta}-e^{-\lambda c}, 1-e^{-\lambda(\theta+c)})}(k) \cdot g(k)dk \\ + I_{(1-e^{-\lambda(\theta+c)}, 1-e^{-2\lambda c}]}(k) \cdot \frac{e^{2\lambda c}}{2 \sinh(\lambda c)} dk & \text{if } 0 \leq \theta \leq c, \\ I_{(0, 1-e^{-\lambda c}]}(k) \cdot \frac{1}{2 \sinh(\lambda c)} dk \\ + I_{(1-e^{-\lambda c}, 1-e^{-2\lambda c}]}(k) \cdot \frac{e^{2\lambda c}}{2 \sinh(\lambda c)} dk & \text{if } -c \leq \theta \leq 0, \\ I_{(0, 1-e^{-\lambda c}]}(k) \cdot \frac{1}{2 \sinh(\lambda c)} dk \\ + I_{(1-e^{-\lambda c}, 1-e^{-2\lambda c}]}(k) \cdot \frac{e^{2\lambda c}}{2 \sinh(\lambda c)} dk & \text{if } \theta \leq -c, \end{cases}$$

where

$$g(k) = \frac{4e^{-\lambda\theta}}{(k + \sqrt{k^2 + 4e^{-\lambda(\theta+c)}})^2 + 4e^{-\lambda(\theta+c)}} + \frac{1}{2 \sinh(\lambda c)}.$$

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References

- [1] R. Albert, and A. -L. Barabási, “Statistical mechanics of complex networks,” *Rev. Modern Phys.* **74**, no.1, 47–97, 2002.
- [2] M. A. Arcones, and E. Giné, “Limit theorems for U -processes,” *Ann. Probab.* **21**, no.3, 1494–1542, 1993.
- [3] P. Billingsley, *Probability and Measure, Third edition, Wiley Series in Probability and Mathematical Statistics*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1995.
- [4] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D. -U. Hwang, “Complex networks: structure and dynamics,” *Phys. Rep.* **424**, no. 4-5, 175–308, 2006.
- [5] M. Boguñá, and R. Pastor-Satorras, “Class of correlated random networks with hidden variables,” *Phys. Rev. E* **68**, 036112, 2003.
- [6] G. Caldarelli, A. Capocci, P. De Los Rios, and M. A. Muñoz, “Scale-free networks from varying vertex intrinsic fitness,” *Phys. Rev. Lett.* **89**, 258702, 2002.
- [7] R. M. Dudley, *Uniform Central Limit Theorems. Cambridge Studies in Advanced Mathematics, 63*. Cambridge University Press, Cambridge, 1999.
- [8] A. Hagberg, D. A. Schult, and P. J. Swart, “Designing threshold networks with given structural and dynamical properties,” *Phys. Rev. E* **74**, 056116, 2006.
- [9] N. Konno, N. Masuda, R. Roy, and A. Sarkar, “Rigorous results on the threshold network model,” *J. Phys. A* **38**, no.28, 6277–6291, 2005.
- [10] N. Masuda, H. Miwa, and N. Konno, “Analysis of scale-free networks based on a threshold graph with intrinsic vertex weights,” *Phys. Rev. E* **70**, 036124, 2004.
- [11] N. Masuda, H. Miwa, and N. Konno, “Geographical threshold graphs with small-world and scale-free properties,” *Phys. Rev. E* **71**, 036108, 2005.
- [12] N. Masuda, and N. Konno, “VIP-club phenomenon: Emergence of elites and masterminds in social networks,” *Social Networks* **28**, no.4, 297–309, 2006.
- [13] A. V. Nagaev, “On estimating the expected number of direct descendants of a particle in a branching process,” *Theory Probab. Appl.* **12**, 314–320, 1967.
- [14] C. A. Najim, and R. P. Russo, “On the number of subgraphs of a specified form embedded in a random graph,” *Methodol. Comput. Appl. Probab.* **5**, no.1, 23–33, 2003.
- [15] M. E. J. Newman, “The structure and function of complex networks,” *SIAM Rev.* **45**, 167–256, 2003.
- [16] G. Peskir, *From Uniform Laws of Large Numbers to Uniform Ergodic Theorems. Lecture Notes Series (Aarhus), 66*. University of Aarhus, Department of Mathematics, Aarhus, 2000.
- [17] R. J. Serfling, *Approximation Theorems of Mathematical Statistics. Wiley Series in Probability and Mathematical Statistics*, John Wiley & Sons, Inc., New York, 1980.
- [18] V. D. P. Servedio, G. Caldarelli, and P. Buttá, “Vertex intrinsic fitness: How to produce arbitrary scale-free networks,” *Phys. Rev. E* **70**, 056126, 2004.
- [19] B. Söderberg, “General formalism for inhomogeneous random graphs,” *Phys. Rev. E* **66**, 066121, 2002.